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The Decomposition Numbers of $Suz(q)$

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An algorithm for the decomposition matrix of $Suz(q)$ is given. By using this algorithm, results of Feit and Chastkofsky, and some other techniques, two rows of the matrix, which correspond to the two imaginary irreducible ordinary characters, are determined. Some other entries are also determined by the same method, also several formulas for the entries of the first column which corresponds to the projective cover of the trivial module are given. As an application all decomposition numbers of $Suz(2^{2n+1})$ for $n \leqslant 3$ are determined. © 1995 Academic Press, Inc.

1. PRELIMINARIES AND NOTATION

In this paper, we are going to discuss the decomposition matrix of the Suzuki groups $Suz(q)$. The following notations will be fixed throughout this paper.

Let F be the algebraic closure of $GF(2)$ and $q = 2^m$, $m = 2n + 1$, where n is a non-negative integer. Let σ be the Frobenius automorphism of F, $a^\sigma = a^2$ for any a in F. σ induces an automorphism of $Sp(4, F)$ which is also denoted by σ. Thus $Sp(4, q)$ consists of the fixed points of σ^m. $Sp(4, F)$ has a graph automorphism τ satisfying

$$\tau^2 = \sigma.$$

$Suz(q)$ is the subgroup of $Sp(4, q)$ consisting of all the elements fixed by τ^m. Denote $Suz(q)$ by G.

$$|G| = q^2(q^2 + 1)(q - 1).$$

See Carter [3] for a detailed discussion of $Suz(q)$ and τ.

Let $r = 2^{n+1}$. There are elements x, y, and z in G, whose orders are $q - 1$, $q + r + 1$, and $q - r + 1$, respectively. The set of the 2′-elements in $G \backslash \{1\}$ is a disjoint union of $\langle x \rangle \backslash \{1\}$, $\langle y \rangle \backslash \{1\}$, $\langle z \rangle \backslash \{1\}$, and their conjugates. Every 2-element in G is conjugate to one of s, t, and t^{-1}, where s and t are

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elements of G and $o(s)=2$, $o(t)=4$. For the conjugate classes of G which will be described below, cf. Suzuki [9].

Let ζ, δ, and ε be linear characters of $\langle x\rangle$, $\langle y\rangle$, and $\langle z\rangle$, respectively, which are defined by

$$\zeta(x)=\exp(2\pi i/q-1),$$
$$\delta(y)=\exp(2\pi i/q+r+1),$$
$$\varepsilon(z)=\exp(2\pi i/q-r+1).$$

In the group ring $\mathbf{Z}[\langle\zeta\rangle]$, $\mathbf{Z}[\langle\delta\rangle]$, and $\mathbf{Z}[\langle\varepsilon\rangle]$, define $\bar{X}_h$, $\bar{Y}_i$, and $\bar{Z}_j$ by

$$\bar{X}_h=\zeta^h+\zeta^{-h},$$
$$\bar{Y}_i=\delta^i+\delta^{-i}+\delta^{iq}+\delta^{-iq},$$
$$\bar{Z}_j=\varepsilon^j+\varepsilon^{-j}+\varepsilon^{jq}+\varepsilon^{-jq},$$

respectively. In particular, $\bar{X}_0=2\cdot 1_{\langle x\rangle}$, $\bar{Y}_0=4\cdot 1_{\langle y\rangle}$, $\bar{Z}_0=4\cdot 1_{\langle z\rangle}$. By definition,

$$\begin{aligned}
\bar{X}_{h_1}=\bar{X}_{h_2} &\Leftrightarrow h_1\equiv \pm h_2 && \pmod{q-1},\\
\bar{Y}_{i_1}=\bar{Y}_{i_2} &\Leftrightarrow i_1\equiv \pm i_2 \text{ or } \pm i_2 q && \pmod{q+r+1},\\
\bar{Z}_{j_1}=\bar{Z}_{j_2} &\Leftrightarrow j_1\equiv \pm j_2 \text{ or } \pm j_2 q && \pmod{q-r+1}.
\end{aligned}$$

Let ψ be the Brauer character of the 4-dimensional natural representation of G. Then we may assume that

$$\psi|_{\langle x\rangle}=\bar{X}_1+\bar{X}_{r-1}$$

by replacing x by a suitable generator of $\langle x\rangle$ if necessary (cf. [8]).

At least one of the eigenvalues of y is a $(q+r+1)$st primitive root of unity, denote it by δ_1. Let f be the characteristic polynomial of y. Then the coefficients of f are all in $GF(q)$. So,

$$f(\delta_1^q)=f(\delta_1)^q=0,$$

i.e., is δ_1^q is also an eigenvalue of y. Since

$$\begin{aligned}(q+r+1,q\pm 1)&\mid((q+r+1)(q-r+1),(q+1)(q-1))\\
&=(q^2+1,q^2-1)=1,\end{aligned}$$

$\delta_1^q\neq\delta_1^{\pm 1}$. By [4, Lemma (3.1)], we may assume

$$\psi|_{\langle y\rangle}=\bar{Y}_1.$$

Similarly, we may assume

$$\psi|_{\langle z\rangle}=\bar{Z}_1.$$

Let $S=\mathbf{Z}/(m)$. For any $k\in S$, define ψ_k by

$$\psi_k=\psi^{\sigma^k}.$$

Then

$$\psi_k|_{\langle x\rangle}=\bar{X}_{2^k}+\bar{X}_{2^k(r-1)},$$
$$\psi_k|_{\langle y\rangle}=\bar{Y}_{2^k},$$
$$\psi_k|_{\langle z\rangle}=\bar{Z}_{2^k}.$$

In particular, $\psi_0=\psi$. (Note. In [4], the notation φ_k is used instead of ψ_k. In fact, $\psi_k=\varphi_{2k}$ for $k=0, ..., n$ and $\psi_{n+k+1}=\varphi_{2k+1}$ for $k=0, ..., n-1$. The labels of the vertices of Scheme 1 have been changed here accordingly.)

For any subset $K\subseteq S$, let

$$\psi_K=\prod_{k\in K}\psi_k.$$

Then the ψ_K's with $K\subseteq S$ are exactly the irreducible Brauer characters of G. For example,

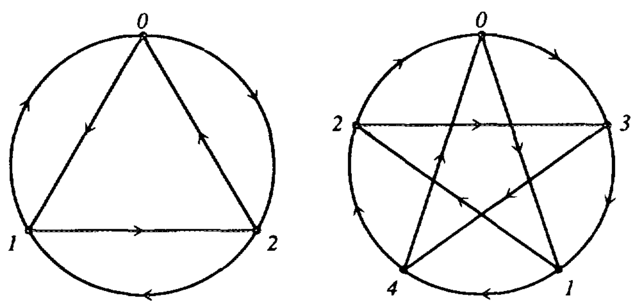
$$\psi_{\varnothing}=1_G,\qquad \text{the trivial character,}$$
$$\psi_S=\Gamma,\qquad \text{the Steinberg character.}$$

Denote by Ψ_K the principal indecomposable character of G, which corresponds to ψ_K.

Scheme 1 plays an important role in Chastkofsky and Feit [4]. The set of vertices of the graph $\mathscr{G}=\mathscr{G}_m$ is S and the set of edges is defined by

$$E=\{(k, k+1), (k, k+n+1)\mid k\in S\}.$$

For example, see Scheme 1.



SCHEME 1

Say two vertices k and l are adjacent if $l=k+n+1$ or $k=l+n+1$. Say a subset K of S is circular, if K' has no adjacent vertices.

The following results are also adopted from [4].

THEOREM (1.1). *For any* $i \in S$,

$$\psi_i^2 = 4 \cdot 1_G + 2\psi_{i+n+1} + \psi_{i+1}.$$

THEOREM (1.2). *Let* $I \subseteq S$. *Then*

$$(\Gamma^2, \psi_I) = \begin{cases} 1, & \textit{if} \quad I=\varnothing \\ 0, & \textit{if} \quad I \neq \varnothing, I \textit{ is not circular} \\ 2^m 4^{-|I'|}, & \textit{if} \quad I \neq S, I \textit{ is circular} \\ 2^m+1, & I=S. \end{cases}$$

THEOREM (1.3). *Let* $K \subseteq S$. *Then*

$$\Psi_K = \Gamma\psi_{K'} - \sum_{\substack{K \subseteq J \\ J-K \text{ circular}}} (\Gamma^2, \psi_{J-K})\, \Gamma\psi_{J'}.$$

The following table of the irreducible ordinary characters of G is adopted from [9]:

χ	$\chi(1)$	$\chi(s)$	$\chi(t), \chi(t^{-1})$	$\chi\|_{\langle x\rangle\setminus\{1\}}$	$\chi\|_{\langle y\rangle\setminus\{1\}}$	$\chi\|_{\langle z\rangle\setminus\{1\}}$
Γ	q^2	0	0	1	-1	-1
X_h	q^2+1	1	1	$\bar{X}_h$	0	0
Y_i	$(q-r+1)(q-1)$	$r-1$	-1	0	$-\bar{Y}_i$	0
Z_j	$(q+r+1)(q-1)$	$-r-1$	-1	0	0	$-\bar{Z}_j$
W_1, W_2	$r(q-1)/2$	$-r/2$	$\pm r\sqrt{-1}/2$	0	1	-1

In Table I, h, i, and j satisfy

$$\begin{aligned} h &\not\equiv 0 \quad (\operatorname{mod} q-1), \\ i &\not\equiv 0 \quad (\operatorname{mod} q+r+1), \\ j &\not\equiv 0 \quad (\operatorname{mod} q-r+1). \end{aligned}$$

2. AN ALGORITHM FOR THE DECOMPOSITION NUMBERS OF G

DEFINITION (2.1). X_0, Y_0, and Z_0 are class functions of G, whose values are given in the following table:

χ	$\chi(1)$	$\chi(s)$	$\chi(t), \chi(t^{-1})$	$\chi\|_{\langle x\rangle\setminus\{1\}}$	$\chi\|_{\langle y\rangle\setminus\{1\}}$	$\chi\|_{\langle z\rangle\setminus\{1\}}$
X_0	q^2+1	1	1	2	0	2
Y_0	$(q-r+1)(q-1)$	$r-1$	-1	0	-4	0
Z_0	$(q+r+1)(q-1)$	$-r-1$	-1	0	0	-4

By the definition of $\bar{X}_0$, $\bar{Y}_0$, and $\bar{Z}_0$ in Section 2,

$$X_0|_{\langle x\rangle\backslash\{1\}}=\bar{X}_0,$$
$$Y_0|_{\langle y\rangle\backslash\{1\}}=\bar{Y}_0,$$
$$Z_0|_{\langle z\rangle\backslash\{1\}}=\bar{Z}_0.$$

It can be checked directly that

$$X_0=1_G+\Gamma,$$
$$Y_0=-1_G+\Gamma-(W_1+W_2),$$
$$Z_0=-1_G+\Gamma+(W_1+W_2),$$

and so, they are orthogonal to each other.

LEMMA (2.2). *For any $i_1, ..., i_k \in S$, if*

$$\prod_{u=1}^{k}(\bar{X}_{2^{i_u}}+\bar{X}_{2^{i_u}(r-1)})=\sum_h a_h\bar{X}_h,$$
$$\prod_{u=1}^{k}\bar{Y}_{2^{i_u}}=\sum_i b_i\bar{Y}_i,$$
$$\prod_{u=1}^{k}\bar{Z}_{2^{i_u}}=\sum_j c_j\bar{Z}_j,$$

then

$$\Gamma\prod_{u=1}^{k}\psi_{i_u}=\sum_h a_hX_h+\sum_i b_iY_i+\sum_j c_jZ_j.$$

Proof. Let $\chi=\sum a_kX_k+\sum b_iY_i+\sum c_jZ_j$. Then it is trivial that

$$\chi|_{\langle x\rangle\cup\langle y\rangle\cup\langle z\rangle\backslash\{1\}}=\Gamma\prod\psi_{i_u}|_{\langle x\rangle\cup\langle y\rangle\cup\langle z\rangle\backslash\{1\}}.$$

Notice that

$$\sum a_h=(2+2)^k/2=2^{2k-1},$$
$$\sum b_i=\sum c_i=4^k/4=2^{2k-2}.$$

Hence

$$\begin{aligned}\chi(1)&=2^{2k-1}(q^2+1)+2^{2k-2}[(q-r+1)(q-1)+(q+r+1)(q-1)]\\&=4^kq^2\\&=\left(\Gamma\prod\psi_{i_u}\right)(1).\end{aligned}$$

Likewise, one can show that

$$\chi(s)=0=\left(\Gamma\prod\psi_{i_u}\right)(s)$$

$$\chi(t)=\chi(t^{-1})=0=\left(\Gamma\prod\psi_{i_u}\right)(t)=\left(\Gamma\prod\psi_{i_u}\right)(t^{-1}). \quad \blacksquare$$

COROLLARY (2.3). *If $K\subset S$ and K' is not circular, then*

$$(\Gamma_{\psi K'}, X_0)=(\Gamma\psi_{K'}, Y_0)=(\Gamma\psi_{K'}, Z_0)=0,$$
$$(\Psi_K, W_1)=(\Psi_K, W_2)=0.$$

Proof. By Theorem (1.3), $\Psi_K=\Gamma\psi_{K'}$. So,

$$(\Gamma\psi_{K'}, \Gamma)=(\Psi_K, \Gamma)=0.$$

Therefore, if $\Gamma\psi_{K'}$ is expanded as in Lemma (2.2), we must have $a_0+b_0+c_0=0$, i.e., $a_0=b_0=c_0=0$.

It follows that

$$0=c_0-b_0=(\Psi_K, W_1)=(\Psi_K, W_2). \quad \blacksquare$$

COROLLARY (2.4). *If $K\subset S$ such that K' is circular and $\Gamma\psi_{K'}$ has been expanded as in Lemma (2.2), then*

$$(\Psi_K, W_1)=(\Psi_K, W_2)=c_0-b_0,$$

and

$$b_0+c_0=2^{2(n-|K|)}, \qquad a_0=2^{2(n-|K|)}+\delta_{S,K'}.$$

Proof. By Theorem (1.3),

$$\Psi_K=\Gamma\psi_{K'}-\sum_{\substack{K\subsetneq J\\ J-K \text{ circular}}}(\Gamma^2, \psi_{J-K})\Psi_J.$$

By Corollary (2.3), for the J's appearing in the sum we have

$$(\Psi_J, W_1)=(\Psi_J, W_2)=0.$$

Hence

$$(\Psi_K, W_i)=(\Gamma\psi_{K'}, W_i)=c_0-b_0, \qquad i=1,2.$$

Since

$$(\Psi_K, 1_G) = \delta_{S,K'},$$
$$(\Gamma\psi_{K'}, 1_G) = a_0 - b_0 - c_0,$$
$$\left(\sum (\Gamma^2, \psi_{J-K})\Psi_J, 1_G\right) = 0,$$
$$(\Psi_K, \Gamma) = 0,$$
$$(\Gamma\psi_{K'}, \Gamma) = a_0 + b_0 + c_0,$$
$$\left(\sum (\Gamma^2, \psi_{J-K})\Psi_J, \Gamma\right) = (\Gamma^2, \psi_{K'}) = 2^{2(n-|K|)+1} + \delta_{S,K'},$$

it follows that

$$a_0 = b_0 + c_0 + \delta_{S,K'},$$
$$a_0 + b_0 + c_0 = 2^{2(n-|K|)+1} + \delta_{S,K'}.$$

Therefore

$$b_0 + c_0 = 2^{2(n-|K|)},$$
$$a_0 = 2^{2(n-|K|)} + \delta_{S,K'}. \quad \blacksquare$$

COROLLARY (2.5). *Let* $K = \{0, 1, ..., n\}'$ *and* a_0, b_0, c_0 *as in Corollary* (2.4). *Then*

$$a_0 = c_0 = 1, \qquad b_0 = 0,$$
$$(\Psi_K, W_1) = (\Psi_K, W_2) = 1.$$

Proof. This is trivial. ∎

3. COMPUTATION OF (Ψ_K, W_i)

LEMMA (3.1).

$$\begin{aligned}\Gamma\psi_{\{0,1,...,l\}} = {} & \sum_{k=1}^{2^l} X_{2k-1} + \sum_{k=1}^{2^l} \sum_{u=1}^{2^l+k-1} X_{ur-2k+1} + \sum_{k=1}^{2^l-1} \sum_{u=1}^{2^l-k} X_{ur+2k-1} \\ & + \sum_{k=1}^{2^l} Y_{2k-1} + \sum_{u=1}^{2^l-1} \sum_{v=1}^{2^l-u} (Y_{2uq-2v+1} + Y_{2uq+2v-1}) \\ & + \sum_{k=1}^{2^l} Z_{2k-1} + \sum_{u=1}^{2^l-1} \sum_{v=1}^{2^l-u} (Z_{2uq-2v+1} + Z_{2uq+2v-1}).\end{aligned}$$

Proof. One can prove it by induction on l. ∎

It is easy to check that

$$\Gamma\psi_{\{0,1,\ldots,2n\}} = \Gamma^2 = 1_G + (q-1)\Gamma + 2^n(W_1 + W_2)$$
$$= +(q+1)\sum_h X_h + (q-r+1)\sum_i Y_i + (q+r+1)\sum_j Z_j.$$

Now, we consider a more general case.

LEMMA (3.2). *If $l \geqslant n$ and*

$$\Gamma\psi_{\{0,1,\ldots,l\}} = \sum_h a_h X_h + \sum_i b_i Y_i + \sum_j c_j Z_j,$$

then

$$b_0 = \tfrac{1}{2}(2^{2(l-n)} - 2^{l-n}), \qquad c_0 = \tfrac{1}{2}(2^{2(l-n)} + 2^{l-n})$$

and

$$(\Gamma\psi_{\{0,1,\ldots,l\}}, W_1) = (\Gamma\psi_{\{0,1,\ldots,l\}}, W_2 = 2^{l-n} = 2^{n-|\{0,1,\ldots,l\}'|}.$$

Proof. By Corollary (2.4), we only have to prove $b_0 = (1/2)(2^{2(l-n)} - 2^{l-n})$. We may assume $n+1 \leqslant l \leqslant 2n-1$.

In Lemma (3.1), $Y_{2k-1} \neq Y_0$ for all k, $1 \leqslant k \leqslant 2^l$, because $0 < 2k-1 \leqslant 2 \cdot 2^{2n-1} < q$. So, we only have to count the Y_0's among the $Y_{2uq-2v+1}$'s and $Y_{2uq+2v-1}$'s.

$Y_{2uq-2v+1} = Y_0 \Leftrightarrow 2uq - 2v + 1 \equiv 0 \pmod{q+r+1} \Leftrightarrow$ for some positive integer i, $2uq - 2v + 1 = (2u-i)(q+r+1)$. Here, i must be an odd number.

Fix a positive odd integer i.

If $2uq - 2v + 1 = (2u-i)(q+r+1)$, then

$$v = i \cdot 2^{2n} + i \cdot 2^n - ru - u + \frac{i+1}{2}.$$

Since

$$1 \leqslant v \leqslant 2^l - u,$$

we have

$$\frac{i \cdot 2^{2n} + i \cdot 2^n + (i+1)/2 - 2^l}{r}$$
$$\leqslant u \leqslant \frac{i \cdot 2^{2n} + i \cdot 2^n + (i-1)/2}{r+1}$$
$$= \frac{i \cdot 2^{n-1}(2^{n+1}+1) + i \cdot 2^{n-1} + (i-1)/2}{2^{n+1}+1}$$
$$= \frac{i \cdot 2^{n-1}(2^{n+1}+1) + (j/2)(2^{n+1}+1) - 2^{n-1} + j/2 - 1}{2^{n+1}+1},$$

i.e.,

$$i\cdot 2^{n-1}+\frac{i}{2}-2^{l-n-1}+\frac{i+1}{2^{n+2}}\leqslant u\leqslant i\cdot 2^{n-1}+\frac{j}{2}-\frac{2^{n-1}-j/2+1}{2^{n+1}+1},$$

where $i=2j-1$. From $u\leqslant 2^l-1$, it follows that

$$i\cdot 2^{n-1}-2^{l-n-1}\leqslant 2^l-1,$$

i.e.,

$$i\leqslant 2^{l-n+1}+\frac{1}{2^{2n-l}}-\frac{1}{2^{n-1}}\leqslant 2^{l-n+1}+\frac{1}{2}.$$

Therefore

$$i\leqslant 2^{l-n+1}-1\leqslant 2^n-1,$$

for i is an odd number ad $l\leqslant 2n-1$. Thus

$$0<\frac{2^{n-1}-j/2+1}{2^{n+1}+1}<\frac{2^{n-1+1}}{2^{n+1}+1}<\frac{1}{2},$$

and

$$0<\frac{i+1}{2^{n+2}}\leqslant\frac{1}{4}.$$

Hence

$$i\cdot 2^{n-1}-2^{l-n-1}+j\leqslant u\leqslant i\cdot 2^{n-1}+\left[\frac{j-1}{2}\right].$$

There are $2^{l-n-1}-j+[(j-1)/2]+1$ such u's for any fixed i. Therefore there are

$$\sum_{2^{l-n-1}-j+[(j-1)/2]+1>0}\left(2^{l-n-1}-j+\left[\frac{j-1}{2}\right]+1\right)$$
$$=2^{l-n-1}+2\sum_{k=1}^{2^{l-n-1}}k=2^{2(l-n-1)}$$

Y_0's among the $Y_{2uq-2v+1}$'s.

Similarly, there are

$$\sum_{2^{l-n-1}-[(j+1)/2]>0}\left(2^{l-n-1}-\left[\frac{j+1}{2}\right]\right)$$
$$=2\sum_{k=1}^{2^{l-n-1}}k=2^{l-n-1}(2^{l-n-1}-1)$$

Y_0's among the $Y_{2uq+2v-1}$'s.

Hence the number of Y_0's, b_0, is

$$2^{l-n-1}(2^{l-n-1}-1)+2^{2(l-n-1)}=\tfrac{1}{2}(2^{2(l-n)}-2^{l-n}). \quad \blacksquare$$

COROLLARY (3.3). *If* $K'=\{0, 1, ..., l\}$ *and* $n \leqslant l \leqslant 2n$, *then*

$$(\Psi_K, W_1)=(\Psi_K, W_2)=2^{n-|K|}.$$

THEOREM (3.4).

$$(\Psi_K, W_1)=(\Psi_K, W_2)=\begin{cases}0, & \text{if } K' \text{ is not circular,}\\ 2^{n-|K|}, & \text{if } K' \text{ is circular.}\end{cases}$$

We start with two observations.

First, for any class functions χ and φ of G, $(\chi^\sigma, \varphi^\sigma)=(\chi, \varphi)$. In particular,

$$(\Gamma\psi_K^\sigma, Y_0)=((\Gamma\psi_K)^\sigma, Y_0^\sigma)=(\Gamma\psi_K, Y_0),$$

i.e.,

$$(\Gamma\psi_{K+1}, Y_0)=(\Gamma\psi_K, Y_0),$$

where $K+1=\{k+1 \mid k \in K\}$. Consequently,

$$(\Gamma\psi_{K+n+1}, Y_0)=(\Gamma\psi_K, Y_0),$$

where $K+n+1=\{k+n+1 \mid k \in K\}$. Notice that $k+n+1$, as a vertex of $\mathscr{G}$, is adjacent to k.

Second, let

$$d_k=|\{K \subseteq S \mid |K|=k, K' \text{ is circular}\}|.$$

Then by the Appendix of [4],

$$\begin{aligned}\sum_k 2^{n-k}\cdot 4^k \cdot d_k &= 2^n \sum_k 2^k d_k\\ &= 2^n \cdot \sum_{K \text{ with no adjacent vertices}} 2^{|K|}=2^n(2^m-1)=\frac{t(q-1)}{2},\end{aligned}$$

which is just the degree of W_1 and W_2.

LEMMA (3.5). *If* $k, k+n+1 \in K$, *but* $k+1 \notin K$, *then*

$$(\Gamma\psi_K, Y_0) \leqslant (\Gamma\psi_{\{k+1\}\cup K\setminus\{k+n+1\}}, Y_0).$$

Proof. Without loss of generality, we may assume that $k=n$ and thus $k+n+1 \equiv 0 \pmod m$, $k+1=n+1$.

Suppose $K=\{k_1, ..., k_s, n, 0\}$. Then $\prod_{k\in K} \bar{Y}_{2^k}$ can be written as a sum of terms of the form

$$\delta^{u_1+\cdots+u_s+v+w},$$

where $u_i \in \{\pm 2^{k_i}, \pm 2^{k_i}q\}$, $i=1, ..., s$, and $v\in\{\pm 2^n, \pm 2^n q\}$, $w\in\{\pm 1, \pm q\}$. $\delta^{u_1+\cdots+u_s+v+w}=1_{\langle y\rangle}$ if and only if

$$u_1+\cdots+u_s+v+w\equiv 0 \qquad (\mathrm{mod}\ q+r+1).$$

Suppose this holds.

If $(v, w)=(2^n q, -1)$, then

$$u_1+\cdots+u_s\equiv 1-2^n q\equiv -2^n \qquad (\mathrm{mod}\ q+r+1)$$

and hence

$$u_1+\cdots+u_s+2^n\equiv 0 \qquad (\mathrm{mod}\ q+r+1),$$

i.e.,

$$\delta^{u_1+\cdots+u_s+2^n}=1_{\langle y\rangle}.$$

By Lemma (2.2), this implies that $(\Gamma\psi_{\{k_1,...,k_s,n\}}, Y_0)\neq 0$. But $n+1$, $0\notin\{k_1, k_s, n\}$, which contradicts Corollary (2.3). Therefore, $(v, w)\neq (2^n q, -1)$.

Similarly, $(v, w)\neq(-2^n q, 1)$, $(2^n, 1)$, $(-2^n, -1)$, $(2^n, q)$, $(-2^n, -q)$, $(2^n q, q)$, $(-2^n q, -q)$.

Because

$$\begin{aligned}
2^n-1&\equiv -2^n q+2^{n+1},\\
-2^n q-1&\equiv 2^n-2^{n+1}q,\\
-2^n+1&\equiv 2^n q-2^{n+1},\\
2^n q+1&\equiv -2^n+2^{n+1}q,\\
-2^n+q&\equiv -2^n q-2^{n+1},\\
-2^n q+q&\equiv -2^n-2^{n+1}q,\\
2^n-q&\equiv 2^n q+2^{n+1},\\
2^n q-q&\equiv 2^n+2^{n+1}q \qquad (\mathrm{mod}\ q+r+1),
\end{aligned}$$

$$\begin{aligned}
(\Gamma\psi_K, Y_0)&=\Big(\prod_{k\in K}\bar{Y}_{2^k}, 1_{\langle y\rangle}\Big)\leqslant\Big(\prod_{k\in\{k+1\}\cup K\setminus\{k+n+1\}}\bar{Y}_k, 1_{\langle y\rangle}\Big)\\
&=\Gamma\psi_{\{k+1\}\cup K\setminus\{k+n+1\}}, Y_0). \quad\blacksquare
\end{aligned}$$

Now, if K' is circular, the above lemma, Corollary (2.4), and Corollary (3.3) imply that

$$(\Psi_K, W_1) = (\Psi_K, W_2) \geqslant 2^{n-|K|}.$$

By the observation in the preceding paragraph, we have now proved the theorem.

4. Preliminary Computations for Cartan Invariants

The results in this section are consequences of Theorem (1.2) and will be used to compute the Cartan invariants.

LEMMA (4.1). *If K is circular, then*

$$(\Psi_K, \Psi_K) = 4^{|K'|}.$$

Proof.

$$\begin{aligned}(\Psi_K, \Psi_K) &= (\Gamma\psi_{K'}, \Gamma_{\psi K'}) = (\Gamma^2, \psi_{K'}^2) \\ &= \left(\Gamma^2, \prod_{k \in K'} (4 \cdot 1_G + 2\psi_{k+n+1} + \psi_{k+1})\right).\end{aligned}$$

If for some $k, l \in K'$, $\psi_{k+n+1} = \psi_{l+1}$, then

$$k + n + 1 \equiv l + 1 \pmod{m}$$

and hence

$$k \equiv k + m = k + n + 1 + n \equiv l + n + 1 \pmod{m},$$

which contradicts the assumption that K is circular.

Therefore we can write

$$\prod_{k \in K'} (4 \cdot 1_G + 2\psi_{k+n+1} + \psi_{k+1}) = 4^{|K'|} \cdot 1_G + \sum_L \psi_L,$$

where the L's are subsets of $\{k+n+1, k+1 \mid k \in K'\}$ with $|L| \leqslant |K'| \leqslant n$.

Since all of the L's are not circular (a circular subset, by definition, must have at least $n+1$ vertices),

$$(\Psi_K, \Psi_K) = (\Gamma^2, 4^{|K'|} \cdot 1_G) = 4^{|K'|},$$

by Theorem (1.2). ∎

Remark. This lemma and Lemma (5.1) below are direct generalizations of (6.4) in [4].

LEMMA (4.2). *Let K be circular and $L=\{u, u+n+1, ..., u+v(n+1)\} \subseteq K$. If for some nonempty $M \subseteq L$, $K' \cup M$ is circular, then*

$$K'=\left\{u+(v+1)(n+1), u+(v+1)(n+1)+1, ..., \right.$$

$$\left. u+(v+1)(n+1)+\left[\frac{m-v}{2}\right]-1\right\},$$

or

$$\left\{u+v(n+1)+1, ..., u+v(n+1)+\left[\frac{m-v-1}{2}\right]-1\right\}.$$

Proof. This is trivial because K' has no adjacent vertices. ∎

LEMMA (4.3). *Let K be a circular subset such that $k, k+n+1 \notin K'$. If*

$$(\Gamma^2, \psi_{K'} \psi_{k+1}) \neq 0,$$

then $K'=\{k+1, ..., k+l\}$ for some l and

$$(\Psi_K, \Psi_{\{k+1\}'})=(\Gamma\psi_{K'}, \Gamma\psi_{k+1})=4.$$

Proof. If $k+1 \notin K'$, then $(\Gamma^2, \psi_{K'}\psi_{k+1})=0$, since k, $k+n+1 \notin K' \cup \{k+1\}$. This contradicts the assumption. So, $k+2 \in K'$ and thus $n+k+2 \notin K'$, for K' has no adjacent vertices. It follows that

$$(\Gamma^2, \psi_{K'}\psi_{k+1})=(\Gamma^2, \psi_{K'\setminus\{k+1\}}(4\cdot 1_G+2\psi_{n+k+2}+\psi_{k+2})).$$

If $K'\setminus\{k+1\}=\varnothing$, then $K'=\{k+1\}$ and the statement has been proved. Otherwise, since $n+k+1$, $k+1 \notin \{n+k+2\} \cup K'\setminus\{k+1\}$, we must have

$$(\Gamma^2, \psi_{K'}\psi_{k+1})=(\Gamma^2, \psi_{K'\setminus\{k+1\}}\psi_{k+2}).$$

Now, the statement can be proved by induction on $|K'|$. ∎

5. COMPUTATIONS OF SOME CARTAN INVARIANTS

In this section, we will determine the Cartan invariants $c_{K,L}=(\Psi_K, \Psi_L)$ where K is circular and $|L'| \leqslant 2$.

LEMMA (5.1). *Let K be circular. Then*

$$(\Psi_K, \Psi_{\{0\}'}) = \begin{cases} 2, & \text{if } K' = \{1, ..., n\} \text{ or } \{n+1, ..., 2n\} \\ 4, & \text{if } K' = \{0, 1, ..., l\}, 0 \leqslant l \leqslant n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. $(\Psi_K, \Psi_{\{0\}'}) = (\Gamma\psi_{K'}, \Gamma\psi_0) = (\Gamma^2, \psi_{K'}\psi_0)$. Suppose it is not zero. If $0 \notin K'$, then $\{0\} \cup K'$ is circular. By Lemma (4.2),

$$K' = \{n+1, ..., 2n\} \quad \text{or} \quad \{1, ..., n\},$$

and hence

$$(\Gamma^2, \psi_{\{0\} \cup K'}) = 2^m \cdot 4^{-n} = 2.$$

If $0 \in K'$, then n, $n+1 \notin K'$ and hence n, $0 \notin \{n+1\} \cup K' \backslash \{0\}$.

$$\begin{aligned}(\Gamma^2, \psi_{K'}\psi_0) &= (\Gamma^2, \psi_{K' \backslash \{0\}}(4 \cdot 1_G + 2\psi_{n+1} + \psi_1)) \\ &= (\Gamma^2, \psi_{K' \backslash \{0\}}(4 \cdot 1_G + \psi_1)).\end{aligned}$$

By Lemma (4.3),

$$K' \backslash \{0\} = \varnothing \quad \text{or} \quad \{1, ..., l\} \text{ for some } l,$$

i.e.,

$$K' = \{0, 1, ..., l\}, \qquad 0 \leqslant l \leqslant n-1,$$

and accordingly

$$(\Psi_K, \Psi_{\{0\}'}) = 4. \quad ∎$$

For $n = 1$, we have

$$(\Psi_{\{0,1\}}, \Psi_2) = 9,$$
$$(\Psi_{\{1,2\}}, \Psi_2) = 10,$$
$$(\Psi_{\{2,0\}}, \Psi_2) = 8.$$

For $n \geqslant 2$, we have the following:

LEMMA (5.2). *Let K be circular. Then*

$$\Psi_K, \Psi_{\{0,n\}'}) = \begin{cases} 2, & \text{if } K' = \{1, ..., n-1\} \\ 4, & \text{if } K' = \{0, n+2, ..., 2n\} \\ 8, & \text{if } K' = \{n+1, ..., 2n\}, \{0, 1, ..., n-1\}, \{n\} \\ & \quad \text{or } \{n, 1, ..., l\}, 1 \leqslant l \leqslant n-1 \\ 16, & \text{if } K' = \{n, n+1, ..., n+l\}, 1 \leqslant l \leqslant n-1 \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA (5.3). *Let K be circular and $1 \leqslant k \leqslant n-1$. Then*

$$(\Psi_K, \Psi_{\{0,k\}'}) = \begin{cases} 2, & \text{if } K' = \{1, ..., k-1, n+k+1, ..., 2n\}, \\ & \quad \text{or } \{1, ..., k-1, k+1, ..., n\} \\ 4, & \text{if } K' = \{0, ..., k-1\}, \\ & \quad \text{or } \{0, ..., k-1, k+1, ..., k+l\},\ 1 \leqslant l \leqslant n-k-1 \\ 8, & \text{if } K' = \{n+1, ..., 2n\}, \{k+1, ..., k+n\}, \{k, ..., n\}, \\ & \quad \{k, ..., n, 1, ..., l\},\ 1 \leqslant l \leqslant k-1, \\ & \quad \text{or } \{0, ..., k-1, n+k+1, ..., n+k+l\},\ 1 \leqslant l \leqslant n-k \\ 16, & \text{if } K' = \{k, ..., n, n+1, ..., n+l\},\ 1 \leqslant l \leqslant k-1, \\ & \quad \text{or } \{0, 1, ..., s, k, k+1, ..., k+t\},\ 0 \leqslant s \leqslant k-1, 0 \leqslant t \leqslant n-k-1 \\ & \text{otherwise.} \end{cases}$$

Both of these two lemmas can be proved by the same method as the proof of Lemma (5.1).

The following two lemmas are about the Cartan invariants $c_{K,L} = (\Psi_K, \Psi_L)$ where K is circular and L' consists of 3 or 4 adjacent vertieces. We omit the proofs.

LEMMA (5.4). *Let K be circular and $n \geqslant 3$. Then*

$$(\Psi_K, \Psi_{\{n,0,n+1\}'}) = \begin{cases} 2, & \text{if } K' = \{0, 2, ..., n-1\} \\ 4, & \text{if } K' = \{n+1, 2, ..., n-1\} \text{ or } \{0, 1, n+3, ..., 2n\} \\ 8, & \text{if } K' = \{1, ..., n-1\} \text{ or } \{n+2, ..., 2n\} \\ 16, & \text{if } K' = \{n\}, \{n, n+2, ..., n+l\},\ 2 \leqslant l \leqslant n-1, \\ & \quad \text{or } \{n, n+1, 2, ..., l\},\ 2 \leqslant l \leqslant n-1 \\ 32, & \text{if } K' = \{n+1, ..., 2n\}\infty, \{0, 1, ..., n-1\}, \\ & \quad \{0, n+2, ..., 2n\}, \\ & \quad \text{or } \{n, 1, ..., l\},\ 1 \leqslant l \leqslant n-1 \\ 48, & \text{if } K' = \{n, n+1\} \\ 64, & \text{if } K' = \{n, n+1, n+2, ..., n+l\},\ 2 \leqslant l \leqslant n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma (5.5) *Let K be circular and $n \geqslant 4$. Then*

$$
(\Psi_K, \Psi_{\{n,0,n+1,1\}'}) = \begin{cases}
2, & \text{if } K' = \{0, n+2, 3, \ldots, n-1\} \\
 & \quad \text{or } \{0, 1, 3, \ldots, n-1\} \\
4, & \text{if } K' = \{n+1, n+2, 3, \ldots, n-1\} \\
 & \quad \text{or } \{0, n+3, \ldots, 2n\} \\
8, & \text{if } K' = \{2, \ldots, n-1\} \\
 & \quad \text{or } \{n+1, n+3, \ldots, 2n\} \\
16, & \text{if } K' = \{1, n+3, \ldots, 2n\}, \\
 & \quad \{n, n+1, n+3, \ldots, n+l\}, 3 \leqslant l \leqslant n-1, \\
 & \quad \text{or } \{n, n+1, n+2, 3, \ldots, l\}, 3 \leqslant l \leqslant n-1 \\
32, & \text{if } K' = \{n+2, \ldots, 2n\}, \{1, \ldots, n-1\}, \\
 & \quad \{n+1, 2, \ldots, n-1\}, \{0, 2, \ldots, n-1\}, \{n\}, \\
 & \quad \{n, 2, \ldots, l\}, 2 \leqslant l \leqslant n-1, \\
 & \quad \text{or } \{n, 1, n+3, \ldots, n+l\}, 3 \leqslant l \leqslant n-1 \\
64, & \text{if } K' = \{n, n+2, \ldots, n+l\}, 2 \leqslant l \leqslant n-1 \\
72, & \text{if } K' = \{0, 1, n+3, \ldots, 2n\} \\
96, & \text{if } K' = \{n, 1\} \\
112, & \text{if } K' = \{n, n+1\} \\
128, & \text{if } K' = \{n+1, \ldots, 2n\}, \{0, n+2, \ldots, 2n\}, \\
 & \quad \{0, 1, \ldots, n-1\}, \\
 & \quad \{n, 1, 2, \ldots, l\}, 2 \leqslant l \leqslant n-1, \\
 & \quad \text{or } \{n, n+1, 2, \ldots, l\}, 2 \leqslant l \leqslant n-1 \\
240, & \text{if } K' = \{n, n+1, n+2\} \\
256, & \text{if } K' = \{n, n+1, n+2, n+3, \ldots, n+l\}, 3 \leqslant l \leqslant n-1, \\
0, & \text{otherwise.}
\end{cases}
$$

Note. The following formulas are direct corollaries of the results of Sections 4 and 5, which will not be used throughout the remainder of this paper.

$$(\Psi_{\varnothing}, \Psi_{\{0\}'}) = \tfrac{4}{3}(2q-1),$$

$$(\Psi_{\varnothing}, \Psi_{\{n,0\}'} = 12(q-2),$$

$$(\Psi_{\varnothing}, \Psi_{\{0,k\}'}) = \tfrac{1}{9}(128q - 112 \cdot 4^{n-k} - 40 \cdot 4^k - 176), \qquad 1 \leqslant k \leqslant n-1,$$

$$(\Psi_{\varnothing}, \Psi_{\{n,0,n+1\}'}) = \tfrac{2}{3}(77q - 292), \qquad n \geqslant 2,$$

$$(\Psi_{\varnothing}, \Psi_{\{n,0,n+1,1\}'} = \tfrac{1}{4}(867q - 6592), \qquad n \geqslant 3.$$

6. Preliminary Computations for Decomposition Numbers

Throughout the remainder of this paper, the following notation will be fixed.

K is a circular subset of S.

L is any subset of $\{0, 1, ..., 2n\}$. L can also be viewed as a subset of S. If $L = \{l_1, ..., l_d\}$, then define X_L, Y_L, and Z_L by

$$X_L = X_l, \qquad Y_L = Y_l, \qquad \text{and} \qquad Z_L = Z_l,$$

respectively, where $l = 2^{l_1} + \cdots + 2^{l_d}$.

Let

$$\begin{aligned} X(i) &= \{\pm 2^i, \pm(2^{n+i+1} - 2^i)\}, && 0 \leqslant i \leqslant n-1, \\ X(n+i) &= \{\pm 2^{n+i}, \pm(2^{n+i} - 2^i)\}, && 0 \leqslant i \leqslant n, \\ Y(i) &= \{\pm 2^i, \pm(2^{n+i+1} + 2^i)\}, && 0 \leqslant i \leqslant n-1, \\ Y(n+i) &= \{\pm 2^{n+i}, \pm(2^{n+i} + 2^i)\}, && 0 \leqslant i \leqslant n, \\ Z(i) &= X(i), \qquad Z(n+i) = X(n+i). \end{aligned}$$

THEOREM (6.1). *The column of the decomposition matrix of G corresponding to Ψ_K consists of* 0's *and* 1's.

Proof. By Lemma (2.2), we can write

$$\Psi_K = \sum_h a_h X_h + \sum_i b_i Y_i + \sum_j c_j Z_j,$$

and

$$\sum_h a_h + \sum_i b_i + \sum_j c_j = \frac{4^{|K'|}}{2} + \frac{4^{|K'|}}{4} + \frac{4^{|K'|}}{4} = 4^{|K'|}.$$

By Corollary (2.3), $a_0 = b_0 = c_0 = 0$. Therefore

$$\sum_h a_h^2 + \sum_i b_i^2 + \sum_j c_j^2 = 4^{|K'|},$$

by Lemma (4.1). This implies that the $4^{|K'|}$ of the a_h's, b_i's, and c_j's are 1 and the others are 0. ∎

COROLLARY (6.2). (i) $(\Psi_K, X_h) = 1$ *if and only if for some* $u_i \in X(i)$,

$$\sum_{i \in K'} u_i \equiv h \qquad (\text{mod } q-1);$$

otherwise $(\Psi_K, X_h) = 0$.

(ii) $(\Psi_K, Y_i)=1$ *if and only if for some* $v_j \in Y(j)$,

$$\sum_{j \in K'} v_j \equiv i \qquad (\bmod\ q+r+1);$$

otherwise $(\Psi_K, Y_i)=0$.

(iii) $(\Psi_K, Z_j)=1$ *if and only if for some* $w_h \in Z(h)$,

$$\sum_{h \in K'} w_h \equiv j \qquad (\bmod\ q-r+1);$$

otherwise $(\Psi_K, Z_j)=0$.

Proof. This is an easy consequence of Theorem (6.1) and Lemma (2.2). ∎

The following corollary explains why we need the information of the Cartan invariants in order to compute the decomposition numbers.

COROLLARY (6.3). *Suppose that L is not circular. If one of* $(\Psi_K, X_L) \neq 0$, $(\Psi_K, Y_L) \neq 0$, *and* $(\Psi_K, Z_L) \neq 0$ *holds, then* $(\Psi_K, \Psi_{L'}) \neq 0$. *If* $(\Psi_K, \Psi_{L'}) = 4^{|L|}$, *then* $(\Psi_K, X_L) = (\Psi_K, Y_L) = (\Psi_K, Z_L) = 1$.

Proof. One only has to notice that X_L, Y_L, and Z_L are constituents of $\Psi_{L'} = \Gamma\psi_L$ by Lemma (2.2). ∎

COROLLARY (6.4). *Suppose* $L = L_1 \cup L_2$, $L_1 \cap L_2 = \varnothing$. $L_1 \subseteq K$ *and* $K \backslash L_1$ *is still circular. If* $(\Psi_K, X_L) = 1$, *then* $(\Psi_{K \backslash L_1}, X_{L_2}) = 1$. *For* Y_L *and* Z_L, *we have similar conclusions.*

Proof. If $\sum_{i \in K'} u_i \equiv \sum_{j \in L} 2^j$ for some $u_i \in X(i)$ or $Y(i)$, then

$$\sum_{i \in K'} u_i - \sum_{j \in L_1} 2^j \equiv \sum_{j \in L_2} 2^j. \quad ∎$$

7. COMPUTATIONS OF SOME DECOMPOSITION NUMBERS

In this section (Ψ_K, X_L), (Ψ_K, Y_L), and (Ψ_K, Z_L) are determined, where $|L| \leqslant 2$ or L consists of 3 or 4 adjacent vertices. Consequently, $(\Psi_\varnothing, X_L)$, $(\Psi_\varnothing, Y_L)$, and $(\Psi_\varnothing, Z_L)$ are obtained by using Theorem (1.3).

THEOREM (7.1). $(\Psi_K, X_1) = 1$ *if and only if*

$$K' = \{1, ..., n\},$$

or

$$\{0, 1, ..., l\}, \qquad 0 \leqslant l \leqslant n-1.$$

$(\Psi_K, X_{r-1}) = 1$, *if and only if*

$$K' = \{n+1, ..., 2n\},$$

or

$$\{0, 1, ..., l\}, \qquad 0 \leqslant l \leqslant n-1.$$

$$(\Psi_\varnothing, X_1) = (\Psi_\varnothing, X_{r-1}) = \frac{2q-1}{3}.$$

$(\Psi_K, Y_1) = 1$ *if and only if*

$$K' = \{0, 1, ..., l\}, \qquad 0 \leqslant l \leqslant n-1.$$

$$(\Psi_\varnothing, Y_1) = \frac{2q-3r+5}{3}.$$

$(\Psi_K, Z_1) = 1$ *if and only if*

$$K' = \{1, ..., n\}, \{n+1, ..., 2n\},$$

or

$$\{0, 1, ..., l\}, \qquad 0 \leqslant l \leqslant n-1.$$

$$(\Psi_\varnothing, Z_1) = \frac{2q+3r-7}{3}.$$

Proof. By (6.1) and (5.1), if one of $(\Psi_K, X_1) \neq 0$, $(\Psi_K, X_{r-1}) \neq 0$, $(\Psi_K, Y_1) \neq 0$, and $(\Psi_K, Z_1) \neq 0$ holds, then

$$K' = \{1, ..., n\}, (n+1, ..., 2n\},$$

or

$$\{0, 1, ..., l\}, \qquad 0 \leqslant l \leqslant n-1,$$

and

$$\begin{aligned}(\Psi_K, X_1) = (\Psi_K, X_{r-1}) = (\Psi_K, Y_1)\\ = (\Psi_K, Z_1) = 1,\end{aligned}$$

when $K' = \{0, 1, ..., l\}$, $0 \leqslant l \leqslant n-1$.

Since

$$-(2+4+\cdots+2^{n-1})+(2^n-1)=1,$$

$$\begin{aligned} -(2^{n+1}+\cdots+2^{2n}) &\equiv -(q-2^{n+1}) \\ &\equiv -(-1)\equiv 1 \qquad (\mathrm{mod}\ q-r+1), \\ -(2^{n+1}+\cdots+2^{2n}) &\equiv -(q-2^{n+1}) \\ &\equiv -(1-2^{n+1})\equiv r-1 \qquad (\mathrm{mod}\ q-1), \end{aligned}$$

by Lemma (6.2),

$$\begin{aligned} (\Psi_{\{1,\dots,n\}'}, Z_1) &= (\Psi_{\{1,\dots,n\}'}, X_1) \\ &= (\Psi_{\{n+1,\dots,2n\}'}, Z_1) = (\Psi_{\{n+1,\dots,2n\}'}, X_{r-1}) = 1. \end{aligned}$$

Notice that

$$\Psi_{\{0\}'} = \Gamma\psi_0 = X_1 + X_{r-1} + Y_1 + Z_1.$$

It follows from Lemma (5.1) that

$$\begin{aligned} (\Psi_{\{1,\dots,n\}'}, X_{r-1}) &= (\Psi_{\{n+1,\dots,2n\}'}, X_1) \\ &= (\Psi_{\{1,\dots,n\}'}, Y_1) = (\Psi_{\{n+1,\dots,2n\}'}, Y_1) = 0. \end{aligned}$$

Using the formula in Theorem (1.3) we obtain that

$$\begin{aligned} (\Psi_{\varnothing}, X_1) &= (\Psi_{\varnothing}, X_r) \\ &= (q+1) - 2^m\cdot 4^{-n} - \sum_{l=0}^{n-1} 2^m\cdot 4^{-(l+1)} \\ &= \frac{1}{3}(2q-1), \\ (\Psi_{\varnothing}, Y_1) &= (q-r+1) - \sum_{l=0}^{n-1} 2^m\cdot 4^{-(l+1)} \\ &= \frac{1}{3}(2q-3r+5), \\ (\Psi_{\varnothing}, Z_1) &= (q+r+1) - 2^m\cdot 4^{-n} - 2^m\cdot 4^{-n} - \sum_{l=0}^{n-1} 2^m\cdot 4^{-(l+1)} \\ &= \frac{1}{3}(2q+3r-7). \quad \blacksquare \end{aligned}$$

We omit the proof of the following 4 theorems.

THEOREM (7.2). *Let n be an integer with $n \geqslant 1$.*
$(\Psi_K, X_{2^n+1}) = 1$ *if and only if*

$$K' = \{0, n+2, ..., 2n\}, \{0, 1, ..., n-1\},$$
$$\{n, 1, ..., l\}, \qquad 1 \leqslant l \leqslant n-1,$$

or

$$\{n, n+1, ..., n+l\}, \qquad 1 \leqslant l \leqslant n-1.$$
$$(\Psi_{\varnothing}, X_{2^n+1}) = \tfrac{5}{6}(q-2).$$

$(\Psi_K, Y_{2^n+1}) = 1$ *if and only if*

$$K' = \{n\},$$

or

$$\{n, n+1, ..., n+l\}, \qquad 1 \leqslant l \leqslant n-1.$$
$$(\Psi_{\varnothing}, Y_{2^n+1}) = \tfrac{1}{3}(2q - 3r + 5).$$

$(\Psi_K, Z_{2^n+1}) = 1$ *if and only if*

$$K' = \{0, n+2, ..., 2n\}, \{n+1, ..., 2n\},$$
$$\{0, 1, ..., n-1\},$$
$$\{n, 1, ..., l\}, \qquad 1 \leqslant l \leqslant n-1,$$

or

$$\{n, n+1, ..., n+l\}, \qquad 1 \leqslant l \leqslant n-1.$$
$$(\Psi_{\varnothing}, Z_{2^n+1}) = \tfrac{1}{6}(5q + 6r - 22).$$

THEOREM (7.3). *Let k be an integer with $1 \leqslant k \leqslant n-1$.*
$(\Psi_K, X_{1+2^k}) = 1$ *if and only if*

$$K' = \{1, ..., k-1, k+1, ..., n\},$$
$$\{0, 1, ..., k-1, k+1, ..., k+l\}, \qquad 1 \leqslant l \leqslant n-k-1,$$
$$\{k+1, k+2, ..., k+n\},$$
$$\{1, ..., l, k, k+1, ..., n\}, \qquad 1 \leqslant l \leqslant k-1,$$
$$\{k, k+1, ..., n+l\}, \qquad 1 \leqslant l \leqslant k-1,$$
$$\{0, ..., s, k, k+1, ..., k+t\}, \qquad 0 \leqslant s \leqslant k-1,\ 1 \leqslant t \leqslant n-k-1.$$
$$(\Psi_{\varnothing}, X_{1+2^k}) = \tfrac{1}{9}(8q - 4^{n-k+1} - 4^k - 47).$$

$(\Psi_K, Y_{1+2^k}) = 1$ *if and only if*

$$K' = \{0, 1, ..., k-1, k+1, ..., k+l\}, \qquad 1 \leqslant l \leqslant n-k-1,$$
$$\{k+1, ..., k+n\}, \{k, ..., n\},$$
$$\{k, k+1, ..., n+l\}, \qquad 1 \leqslant l \leqslant k-1,$$
$$\{0, ..., s, k, k+1, ..., k+t\}, \qquad 0 \leqslant s \leqslant k-1, 0 \leqslant t \leqslant n-k-1.$$
$$(\Psi_\varnothing, Y_{1+2^k}) = \tfrac{1}{9}(8q - 9r - 4^{n-k+1} - 4^{k+1} + 19).$$

$(\Psi_K, Z_{1+2^k}) = 1$ *if and only if*

$$K' = \{1, ..., k-1, k+1, ..., n\},$$
$$\{0, 1, ..., k-1, k+1, ..., k+l\}, \qquad 1 \leqslant l \leqslant n-k-1,$$
$$\{n+1, ..., 2n\}, \{k+1, ..., k+n\},$$
$$\{1, ..., l, k, k+1, ..., n\}, \qquad 1 \leqslant l \leqslant k-1,$$
$$\{k, k+1, ..., n+l\}, \qquad 1 \leqslant l \geqslant k-1,$$
$$\{0, ..., s, k, k+1, ..., k+t\}, \qquad 0 \leqslant s \leqslant k-1, 0 \leqslant t \leqslant n-k-1.$$
$$(\Psi_\varnothing, Z_{1+2^k}) = \tfrac{1}{9}(8q + 9r - 4^{n-k+1} - 4^k - 65).$$

THEOREM (7.4). *Let n be an integer with* $n \geqslant 2$.
$(\Psi_K, X_{2^n+1+2^{n+1}}) = 1$ *if and only if*

$$K' = \{0, 1, n+3, ..., 2n\},$$
$$\{n, n+2, ..., n+l\}, \qquad 2 \leqslant l \leqslant n-1,$$
$$\{n, n+1, 2, ..., l\}, \qquad 2 \leqslant l \leqslant n-1,$$
$$\{n+1, ..., 2n\}, \{0, 1, ..., n-1\},$$
$$\{0, n+2, ..., 2n\},$$
$$\{n, n+1, n+2, ..., n+l\}, \qquad 2 \leqslant l \leqslant n-1.$$
$$(\Psi_\varnothing, X_{2^n+1+2^{n+1}}) = \tfrac{1}{8}(7q - 24).$$

$(\Psi_K, Y_{2^n+1+2^{n+1}}) = 1$ *if and only if*

$$K' = \{n+1, ..., 2n\},$$
$$\{n, 1, ..., l\}, \qquad 1 \leqslant l \leqslant n-1,$$
$$\{n, n+1\},$$
$$\{n, n+1, n+2, ..., n+l\}, \qquad 2 \leqslant l \leqslant n-1.$$
$$(\Psi_\varnothing, Y_{2^n+1+2^{n+1}}) = \tfrac{1}{6}(5q - 6r + 2).$$

$(\Psi_K, Z_{2^n+1+2^{n+1}}) = 1$ *if and only if*

$$K' = \{n+3, ..., 2n, 0, 1\}, \{n+2, ..., 2n\},$$
$$\{n, n+2, ..., n+l\}, \quad 2 \leqslant l \leqslant n-1,$$
$$\{n, n+1, 2, ..., l\}, \quad 2 \leqslant l \leqslant n-1,$$
$$\{0, 1, ..., n-1\}, \{0, n+2, ..., 2n\},$$
$$\{n, n+1, n+2, ..., n+l\}, \quad 2 \leqslant l \leqslant n-1.$$
$$(\Psi_\varnothing, Z_{2^n+1+2^{n+1}}) = \tfrac{1}{8}(7q+8r-72).$$

THEOREM (7.5). *Let n be an integer with* $n \geqslant 4$.
$(\Psi_K, X_{2^n+1+2^{n+1}+2}) = 1$ *if and only if*

$$K' = \{0, n+2, 3, ..., n-1\},$$
$$\{n, n+1, n+2, 3, ..., l\}, \quad 3 \leqslant l \leqslant n-1,$$
$$\{n, 1, n+3, ..., n+l\}, \quad 3 \leqslant l \leqslant n-1,$$
$$\{0, 1, n+3, ..., 2n\},$$
$$\{n+1, n+2, ..., 2n\},$$
$$\{0, n+2, ..., 2n\},$$
$$\{0, 1, ..., n-1\},$$
$$\{n, 1, 2, ..., l\}, \quad 2 \leqslant l \leqslant n-1,$$
$$\{n, n+1, 2, ..., l\}, \quad 2 \leqslant l \leqslant n-1,$$
$$\{n, n+1, n+2, n+3, ..., n+l\}, \quad 3 \leqslant l \leqslant n-1.$$
$$(\Psi_\varnothing, X_{2^n+1+2^{n+1}+2}) = \tfrac{1}{96}(89q-928).$$

$(\Psi_K, Y_{2^n+1+2^{n+1}+2}) = 1$ *if and only if*

$$K' = \{n, n+2, ..., n+l\}, \quad 2 \leqslant l \leqslant n-1,$$
$$\{n, n+1\},$$
$$\{n+1, ..., 2n\},$$
$$\{0, n+2, ..., 2n\},$$
$$\{n, n+1, n+2\},$$
$$\{n, n+1, n+2, n+3, ..., n+l\}, \quad 3 \leqslant l \leqslant n-1.$$
$$(\Psi_\varnothing, Y_{2^n+1+2^{n+1}+2}) = \tfrac{1}{6}(5q-6r+2).$$

$(\Psi_K, Z_{2^n+1+2^{n+1}+2}) = 1$ *if and only if*

$$\begin{aligned}
K' = &\{0, n+2, 3, ..., n-1\},\\
&\{1, n+3, ..., 2n\},\\
&\{n, n+1, n+2, 3, ..., l\}, \qquad 3 \leqslant l \leqslant n-1,\\
&\{n, 1, n+3, ..., n+l\}, \qquad 3 \leqslant l \leqslant n-1,\\
&\{0, 1, n+3, ..., 2n\},\\
&\{0, n+2, ..., 2n\},\\
&\{0, 1, ..., n-1\},\\
&\{n, 1, 2, ..., l\}, \qquad 2 \leqslant l \leqslant n-1,\\
&\{n, n+1, 2, ..., l\}, \qquad 2 \leqslant l \leqslant n-1,\\
&\{n, n+1, n+2, n+3, ..., n+l\}, \qquad 3 \leqslant l \leqslant n-1.\\
&(\Psi_\varnothing, Z_{2^n+1+2^{n+1}+2}) = \tfrac{1}{96}(89q + 96r - 1504).
\end{aligned}$$

APPENDIX: EXAMPLES

Suz(2)

The first member of the series of Suzuki groups is a solvable group of order 20 with the following relations:

$$t^4 = y^5 = 1, \qquad t^{-1}yt = y^3.$$

So, everything is trivial.

The decomposition matrix:

K:	$\varnothing$	0
1_G	1	
Y	1	
W_1	1	
W_2	1	
Γ		1

The Cartan matrix:

4	0
0	1

The projective cover of the trivial module is uniserial.

$Suz(2^3)$

The decomposition matrix:

K:	$\emptyset$	0	1	2	1 2	2 0	0 1	0 1 2
1_G	1							
X_1	5	2	2	3	1	0	1	
X_2	5	3	2	2	1	1	0	
X_4	5	2	3	2	0	1	1	
Y_1	3	1	1	2	1			
Y_2	3	2	1	1		1		
Y_4	3	1	2	1			1	
Z	7	3	3	3	1	1	1	
W_1	2	1	1	1				
W_2	2	1	1	1				
Γ								1

The Cartan matrix:

160	72	72	72	20	20	20	
72	34	32	32	9	10	8	
72	32	34	32	8	9	10	
72	32	32	34	10	8	9	
20	9	8	10	4	2	2	
20	10	9	8	2	4	2	
20	8	10	9	2	2	4	
							1

Suz(2^5)
The decomposition matrix:

K:	$\emptyset$	0	1	2	3	4	01	12	23	34	40	02	13	24	30	41	134	240	301	412	023	234	340	401	012	123	1234	2340	3401	4012	0123	01234
1_G	1																															
X_1	21	7	5	9	8	6	1	2	2	2	1	1	1	4	3	1	0	0	0	2	1	1	1	0	0	0	1					
X_2	21	6	7	5	9	8	1	1	2	2	2	1	1	1	4	3	1	0	0	0	2	0	1	1	0	0		1				
X_4	21	8	6	7	5	9	2	1	1	2	2	3	1	1	1	4	2	1	0	0	0	0	0	1	1	0			1			
X_8	21	9	8	6	7	5	2	2	1	1	2	4	3	1	1	1	0	2	1	0	0	0	0	0	1	1				1		
X_{15}	21	5	9	8	6	7	2	2	2	1	1	1	4	3	1	1	0	0	2	1	0	1	0	0	0	1					1	
X_7	21	10	7	8	8	6	3	2	2	2	3	3	1	4	1	0	0	0	0	2	0	1	0	0	1	0	1					
X_{14}	21	6	10	7	8	8	3	3	2	2	2	0	3	1	4	1	0	0	0	0	2	0	1	0	0	1		1				
X_3	21	8	6	10	7	8	2	3	3	2	2	1	0	3	1	4	2	0	0	0	0	1	0	1	0	0			1			
X_6	21	8	8	6	10	7	2	2	3	3	2	4	1	0	3	1	0	2	0	0	0	0	1	0	1	0				1		
X_{12}	21	7	8	8	6	10	2	2	2	3	3	1	4	1	0	3	0	0	2	0	0	0	0	1	0	1					1	
X_5	25	9	7	9	8	9	2	2	2	2	3	2	2	3	3	3	1	0	0	1	1	1	1	1	0	1						
X_{10}	25	9	9	7	9	8	3	2	2	2	2	3	2	2	3	3	1	1	0	0	1	1	1	1	1	0						
X_{11}	25	8	9	9	7	9	2	3	2	2	2	3	3	2	2	3	1	1	1	0	0	0	1	1	1	1						
X_9	25	9	8	9	9	7	2	2	3	2	2	3	3	3	2	2	0	1	1	1	0	1	0	1	1	1						
X_{13}	25	7	9	8	9	9	2	2	2	3	2	2	3	3	3	2	0	0	1	1	1	1	1	0	1	1						
Y_1	15	6	3	8	6	3	1	2	2	1	1	1	0	4	1	0	0	0	0	2	0	1	0	0	0	0	1					
Y_2	15	3	6	3	8	6	1	1	2	2	1	0	1	0	4	1	0	0	0	0	2	0	1	0	0	0		1				
Y_4	15	6	3	6	3	8	1	1	1	2	2	1	0	1	0	4	2	0	0	0	0	0	0	1	0	0			1			
Y_8	15	8	6	3	6	3	2	1	1	1	2	4	1	0	1	0	0	2	0	0	0	0	0	0	1	0				1		
Y_{16}	15	3	8	6	3	6	2	2	1	1	1	0	4	1	0	1	0	0	2	0	0	0	0	0	0	1					1	
Y_3	19	6	6	7	7	7	1	2	2	2	2	1	1	2	3	3	1	0	0	0	1	1	1	1	0	0						
Y_6	19	7	6	6	7	7	2	1	2	2	2	3	1	1	2	3	1	1	0	0	0	0	1	1	1	0						
Y_{12}	19	7	7	6	6	7	2	2	1	2	2	3	3	1	1	2	0	1	1	0	0	0	0	1	1	1						
Y_{11}	19	7	7	7	6	6	2	2	2	1	2	2	3	3	1	1	0	0	1	1	0	1	0	0	1	1						
Y_7	19	6	7	7	7	6	2	2	2	2	1	1	2	3	3	1	0	0	0	1	1	1	1	0	0	1						
Z_1	27	10	9	9	10	9	3	2	2	3	3	3	2	4	3	1	0	0	0	2	1	1	1	0	1	0	1					
Z_2	27	9	10	9	9	10	3	3	2	2	3	1	3	2	4	3	1	0	0	0	2	0	1	1	0	1		1				
Z_4	27	10	9	10	9	9	3	3	3	2	2	3	1	3	2	4	2	1	0	0	0	1	0	1	1	0			1			
Z_8	27	9	10	9	10	9	2	3	3	3	2	4	3	1	3	2	0	2	1	0	0	0	1	0	1	1				1		
Z_9	27	9	9	10	9	10	2	2	3	3	3	2	4	3	1	3	0	0	2	1	0	1	0	1	0	1					1	
Z_5	31	10	10	10	10	10	2	2	2	2	2	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1						
W_1	4	2	2	2	2	2	1	1	1	1	1																					
W_2	4	2	2	2	2	2	1	1	1	1	1																					
Γ																																1

$Suz(2^7)$
The decomposition matrix:

K:	$\emptyset$	0	01	02	012	03	034	013	0135	023	1456	1246	2456	1256	12456	3456	13456	23456	123456	0123456
1_G	1																			
X_1	85	29	6	6	1	6	3	1	1	1						1		1	1	
X_2	85	22	6	5	1	1	8	0	0	0	1									
X_4	85	24	6	6	1	3	0	1	0	0	2		1							
X_8	85	32	8	8	2	10	0	3	3	3			2	4	2					
X_{16}	85	33	8	8	2	16	0	4	4	4										
X_{32}	85	21	8	8	2	1	0	0	0	0										
X_{63}	85	22	5	8	2	4	0	1	1	0						1				
X_3	101	30	7	6	1	6	1	1	2	1	2					1	1	1		
X_6	101	30	7	7	1	6	6	1	1	1		1	2	3	1					
X_{12}	101	32	8	8	2	15	0	4	4	4	2									
X_{24}	101	31	8	8	2	6	0	1	1	1		2	2	2						
X_{48}	101	34	9	8	2	14	0	4	4	4										
X_{31}	101	27	8	7	2	6	1	2	1	1						1		1		
X_{62}	101	27	6	9	2	2	7	0	0	0										
X_5	105	31	7	7	1	6	4	1	2	1	1	1	2	1		1	1			
X_{10}	105	32	7	8	2	13	2	3	4	4	1	1	2	1						
X_{20}	105	33	8	9	2	13	0	3	4	4		2	1	1						
X_{40}	105	31	8	8	2	12	0	3	3	4		2								
X_{47}	103	35	8	9	2	11	1	2	2	4						1				
X_{33}	105	28	9	7	2	5	5	2	1	0						1				
X_{66}	105	28	6	8	2	6	5	1	2	1	1					1	1			
X_7	85	32	8	8	2	6	0	1	1	1			2	4	2	1				
X_{14}	85	32	8	8	2	16	0	4	4	4										
X_{28}	85	23	8	8	2	1	0	0	0	0										
X_{56}	85	26	7	8	2	4	0	1	1	0										
X_{15}	85	36	10	10	3	10	1	3	3	3						1		1	1	
X_{30}	85	22	10	7	3	0	8	0	0	0										
X_{60}	85	24	6	10	3	1	0	0	0	0	2									

K:	∅	0	01	02	012	03	034	013	035	023	1456	1246	2456	1256	12456	3456	13456	23456	123456	0123456
X_9	105	33	7	8	2	7	4	1	1	1		1	2	3	1	1				
X_{18}	105	33	9	8	2	15	1	4	4	4	1									
X_{36}	105	29	8	7	2	6	0	1	2	1		2	1	1		1				
X_{55}	105	31	7	8	2	12	1	3	3	3		1								
X_{17}	105	35	8	9	2	10	1	2	2	3						1		1		
X_{34}	105	27	9	7	2	3	7	1	1	0		1								
X_{59}	105	30	7	8	2	8	1	2	2	3	2					1	1			
X_{11}	105	34	9	8	2	8	0	2	2	1	2		1			1	1			
X_{22}	105	33	9	9	3	10	4	3	3	3		1	2	3	1	1				
X_{44}	105	32	8	8	2	15	0	4	4	4	1									
X_{39}	105	31	8	9	2	4	0	1	0	0		2	1	1						
X_{49}	105	34	9	8	2	12	1	3	3	4						1				
X_{29}	105	29	8	7	2	7	3	2	2	2						1		1		
X_{58}	105	29	7	9	2	2	7	0	0	0	1									
X_{13}	105	35	8	9	2	9	3	2	3	2	1					1		1		
X_{26}	105	29	10	7	2	3	7	1	0	0	1	1								
X_{52}	105	34	9	10	3	10	0	3	2	3	2		1			1	1			
X_{23}	105	32	9	8	2	10	2	3	3	3		1	2	3	1					
X_{46}	105	33	8	8	2	15	0	4	4	4										
X_{35}	105	29	8	8	2	4	0	1	1	0		2				1				
X_{57}	105	30	7	8	2	10	2	2	2	4						1				
X_{19}	109	34	9	8	2	10	2	3	2	3	1	1				1	1			
X_{38}	109	31	9	8	2	9	4	3	2	3	1	1	2	1		1				
X_{51}	109	33	8	8	2	13	1	3	4	4	1	1								
X_{25}	109	31	8	7	2	9	1	3	2	3		2	1	1		1				
X_{50}	109	34	8	9	2	12	2	3	3	4		1				1				
X_{27}	109	30	8	7	2	8	2	2	2	3	1					1				
X_{54}	109	32	7	9	2	7	5	1	2	1	1	1				1				
X_{21}	105	33	8	8	2	10	0	3	3	3		2	2	2		1				
X_{42}	105	31	8	8	2	14	1	4	4	4										
X_{43}	105	33	8	9	2	8	0	2	3	1						1				
X_{41}	105	32	10	8	2	6	5	2	2	0		1				1				
X_{45}	105	35	10	10	3	10	3	3	2	3	1					1	1	1		
X_{37}	105	30	9	9	3	3	6	1	0	0	1	1	2	1						
X_{53}	105	32	8	8	2	13	0	3	4	4	2		1							

K:	∅	0	01	02	012	03	034	013	035	023	1456	1246	2456	1256	12456	3456	13456	23456	123456	0123456
Y_1	71	28	6	6	1	6	1	1	1	1						1	0	1	1	
Y_2	71	15	6	3	1	0	8	0	0	0										
Y_4	71	18	3	6	1	1	0	0	0	0	2									
Y_8	71	28	6	6	1	6	0	1	1	1			2	4	2					
Y_{16}	71	32	8	8	2	16	0	4	4	4										
Y_{32}	71	15	8	6	2	0	0	0	0	0										
Y_{64}	71	18	3	8	2	2	0	0	0	0										
Y_3	91	28	6	6	1	6	3	1	2	1	1					1	1	1		
Y_6	91	25	7	6	1	3	6	1	0	0	1	1	2	1						
Y_{12}	91	30	7	8	2	13	0	3	4	4	2		1							
Y_{24}	91	29	8	7	2	10	0	3	3	3		2	2	2						
Y_{48}	91	31	8	8	2	14	0	4	4	4										
Y_{49}	91	25	8	7	2	5	0	2	1	0						1				
Y_{47}	91	25	6	8	2	4	5	1	1	0						1				
Y_5	91	28	6	6	1	6	0	1	2	1	2		1			1	1			
Y_{10}	91	29	7	7	2	10	2	3	3	3		1	2	3	1					
Y_{20}	91	31	8	8	2	15	0	4	4	4										
Y_{40}	91	25	8	7	2	3	0	1	1	0		2								
Y_{65}	91	30	7	8	2	10	1	2	2	4						1				
Y_{15}	91	28	8	7	2	6	3	2	1	1						1		1		
Y_{30}	91	22	6	7	2	1	7	0	0	0	1									
Y_7	91	28	7	6	1	6	4	1	1	1		1	2	3	1	1				
Y_{14}	91	31	8	8	2	15	0	4	4	4	1									
Y_{28}	91	25	8	6	2	3	0	1	0	0		2	1	1						
Y_{56}	91	30	7	8	2	12	0	3	3	4										
Y_{33}	91	29	8	7	2	7	1	2	2	2						1		1		
Y_{66}	91	22	7	7	2	1	7	0	0	0										
Y_{13}	91	28	6	8	2	6	0	1	2	1	2					1	1			
Y_{11}	95	28	7	7	2	6	4	1	2	1	1	1	2	1		1				
Y_{22}	95	30	7	8	2	13	1	3	4	4	1	1								
Y_{44}	95	29	8	8	2	9	0	3	2	3		2	1	1		1				
Y_{57}	95	30	8	8	2	12	1	3	3	4		1								
Y_{31}	95	29	8	7	2	8	1	2	2	3						1				
Y_{62}	95	25	7	7	2	4	5	1	1	0		1				1				
Y_{21}	95	29	7	8	2	8	2	2	2	3	1					1	1			
Y_{29}	99	29	8	7	2	9	2	3	2	3	1	1				1				

K :	$\emptyset$	0	0 1	0 2	0 1 2	0 3	0 3 4	0 1 3	0 3 5	0 2 3	1 4 5 6	1 2 4 6	2 4 5 6	1 2 5 6	1 2 4 5 6	3 4 5 6	1 3 4 5 6	2 3 4 5 6	1 2 3 4 5 6	0 1 2 3 4 5 6
Z_1	99	36	10	10	3	10	3	3	3	3						1		1	1	
Z_2	99	29	10	9	3	1	8	0	0	0	1									
Z_4	99	30	9	10	3	3	0	1	0	0	2		1							
Z_8	99	36	10	10	3	10	0	3	3	3			2	4	2	1				
Z_{16}	99	33	8	8	2	16	0	4	4	4										
Z_{32}	99	29	8	10	2	2	0	0	0	0										
Z_{49}	99	30	9	8	2	6	0	2	2	0						1				
Z_3	115	36	10	10	3	10	1	3	2	3	2					1	1	1		
Z_6	115	35	9	10	3	6	6	1	1	1		1	2	3	1					
Z_{12}	115	34	9	8	2	15	0	4	4	4	2									
Z_{24}	115	35	8	9	2	7	0	1	1	1		2	2	2		1				
Z_{48}	115	34	9	8	2	14	1	4	4	4										
Z_{17}	115	35	8	9	2	9	1	2	3	2						1		1		
Z_{34}	115	34	10	9	2	4	7	1	1	0		1								
Z_5	119	36	9	10	3	10	4	3	2	3	1	1	2	1		1	1			
Z_{10}	119	35	9	9	2	13	2	3	4	4	1	1	2	1						
Z_{20}	119	35	8	8	2	13	0	3	4	4		2	1	1						
Z_{40}	119	35	8	9	2	12	1	3	3	4		2				1				
Z_{33}	119	35	8	9	2	11	2	2	2	4						1				
Z_{47}	119	35	9	9	2	8	5	2	3	1	1					1				
Z_{19}	119	35	10	8	2	8	5	2	2	1	1	1				1	1			
Z_9	119	36	9	9	2	10	4	3	3	3		1	2	3	1	1				
Z_{18}	119	34	8	8	2	15	1	4	4	4	1									
Z_{36}	119	35	8	10	2	7	0	1	2	1		2	1	1		1				
Z_{41}	119	35	9	8	2	12	2	3	3	4		1				1				
Z_{31}	119	36	8	9	2	10	3	2	2	3	1					1		1		
Z_{51}	119	34	9	9	2	4	7	1	0	0	1	1								
Z_{11}	119	36	10	8	2	10	1	3	2	3	2		1			1	1			
W_1	8	4	2	2	1															
W_2	8	4	2	2	1															
Γ																				1

Note. Other entries are determined by

$$(X_{2i}, \Psi_{K+1}) = (X_i, \Psi_K),$$
$$(Y_{2i}, \Psi_{K+1}) = (Y_i, \Psi_K),$$
$$(Z_{2i}, \Psi_{K+1}) = (Z_i, \Psi_K),$$

where $K+1 = \{k+1 \mid k \in K\}$.

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